Static Electromagnetic Fields of an Underwater Pipeline

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Abstract

Analytical expressions for the static electromagnetic fields due to a pipeline are given for the permanent and induced magnetic fields as well as the corrosion related electric and magnetic fields. Randomly oriented dipoles are used for the permanent magnetic field while the induced field is calculated using the solution of a hollow cylinder in a uniform field. For the corrosion related electric field, expressions for a line source with multiple point current sources representing sacrificial anodes is derived. Using current monopole theory the electric model is equivalently represented with current filaments to provide expressions for the corrosion related magnetic field. A comparison of the magnetic field strengths is provided and the fall-off rates predicted by the models is shown. Maple's Vector Calculus library is used extensively in the derivations and Maple worksheets with the full solution details are provided in the appendix.

1 Introduction

Pipelines can considerably reduce the cost of transporting oil and gas over long distances. Their presence in the earth's magnetic field and in a conducting medium such as the earth or ocean creates detectable magnetic and electric fields. To quantify the strength of these disturbances analytical solutions of the electric and magnetic fields around a pipeline are derived and compared. The problem divides into four distinct cases, the permanent magnetic field, the induced magnetic field, the corrosion related electric field and the corrosion related magnetic field.

The calculation of the static electric and magnetic fields created when a steel structure wi. an arbitrary geometry is placed in a magnetic field and in an conducting medium are, in this supercomputer age, usually analyzed using modern finite element and boundary element procedures. Such solutions, however, have their limitations particularly when dealing with large geometries over long distances. Analytically solutions, on the other hand, are obtainable for only a limited set of geometries but can be structured from the outset to deal with large geometries. The electric and magnetic fields existing around

a long submerged pipeline represents a case where analytical solutions can be derived. Maple's Vector Calculus library significantly eases the algebraical burden required for the derivations and Maple worksheets with the full solution details are provided as an appendix.

1.1 Magnetostatic review

The permanent and induced magnetic fields of a pipeline fall into the category of a static magnetic field [1]. When the fields are constant and assuming, for the moment, no electrical currents are present, Maxwell's equations simplify to the following two vector field equations

$$\boldsymbol{\nabla} \times \boldsymbol{H} = 0 \tag{1}$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{2}$$

where $\boldsymbol{H}$ is the magnetic field intensity in (A/m), $\boldsymbol{B}$ is the magnetic field in (T), and the curl ($\boldsymbol{\nabla}\times$) and divergence ($\boldsymbol{\nabla}\cdot$) operators are defined in their usual manner. Any vector field with a curl of zero can be expressed as the gradient of a scalar potential and therefore the magnetic intensity can be written as

$$\boldsymbol{H} = -\boldsymbol{\nabla}\psi \tag{3}$$

where $\boldsymbol{\nabla}$ is the gradient operator and ψ is the scalar magnetic potential. For an isotropic medium we can write $\boldsymbol{B} = \mu\boldsymbol{H}$ where μ is the magnetic permeability of the medium which leads to Laplace's equation for magnetostatics

$$\nabla^2\psi = 0. \tag{4}$$

To model the pipeline's permanent magnetic field we will satisfy (4) using a distribution of point magnetic dipoles while for the pipeline's induced field (4) is solved using separation of variables in a cylindrical coordinate system.

1.2 Electromagnetostatic review

To protect a pipeline from corroding, coatings and electrical corrosion protection systems are used. When the power grid is accessible active protection is used while in remote locations copper or aluminum sacrificial anodes are applied. In either event a steady electric current will be developed to protect the pipeline and with the flow of electrical current we must modify (1) such that

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} \tag{5}$$

where $\boldsymbol{J}$ is the current density in (A/m^2). For an isotropic material the current density relates to the electric field by Ohm's law

$$\boldsymbol{J} = \sigma\boldsymbol{E} \tag{6}$$

where $\boldsymbol{E}$ is the electric field in (V/m) and σ is the conductivity of the medium in (Ω^{-1}/m). Static electric fields are governed by the following two equations

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \tag{7}$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0. \tag{8}$$

Thus $\boldsymbol{E}$ can be expressed as the gradient of a scalar potential

$$\boldsymbol{E} = -\boldsymbol{\nabla} V \tag{9}$$

where V is the electric potential in (V) and

$$\nabla^2 V = 0, \tag{10}$$

Laplace's equation for the electric potential for static electric fields.

To model the electrical characteristics of a pipeline we will use point and line current sources that satisfy (10). To determine the corrosion related magnetic field we use an alternate but equivalent model based on current monopole theory [2].

2 Permanent magnetic field

During construction of a pipeline individual sections of pipe are welded together to form a continuous length. Each pipe section will in general have a unique permanent magnetization which can vary significantly depending on its magnetic history. Measurements made at the Sable Island gas pipeline verify this to be the case [3] and measured magnetic field data was modelled by a series of prolate spheroids and the magnetizations were found to vary randomly along the pipe. For the present analysis it is sufficient to treat each section as a magnetic dipole with an arbitrary orientation.

The scalar magnetic potential of a point magnetic dipole can be represented as

$$\psi = \frac{1}{4\pi} \boldsymbol{m} \cdot \boldsymbol{\nabla}\left(\frac{1}{R}\right) \tag{11}$$

where $\boldsymbol{m}$ is the vector dipole moment in (Am) and R is the distance from the point dipole source to the observation point. When N magnetic dipoles are superimposed the potential becomes

$$\psi = \sum_{n=1}^{N} \boldsymbol{m}_n \cdot \boldsymbol{\nabla}\left(\frac{1}{R_n}\right) \tag{12}$$

where $\boldsymbol{m}_n$ is the magnetic moment of the dipole located R_n from the observation point. The magnetic intensity is calculated as $\boldsymbol{H} = -\boldsymbol{\nabla}\psi$ and the magnetic field as $\boldsymbol{B} = \mu \boldsymbol{H}$. We will investigate the relative strength of the permanent magnetization in the summary of results.

3 Induced magnetic field

Unlike the permanent field, the induced magnetic field of a pipeline will be more uniform since the magnetic permeability of each section will be more or less constant. The Sable Island gas pipeline runs approximately perpendicular to the direction of the earth's magnetic field and the earth's field over the length of the pipe is nearly constant. This situation can be represented by the case of a long hollow cylinder in a uniform magnetic field[1]. This particular problem is included in Stratton's problem set [4, p265] and is reviewed here to demonstrate the power of Maple to solve such problems.

We introduce a coordinate system where x is horizontal, y is vertical and z is along the pipeline centerline, and a cylindrical coordinates (r, ϕ, z) such that $x = r\cos(\phi)$, $y = r\sin(\phi)$, and $z = z$. For the present problem the inner radius of the pipeline is defined as a and its outer radius as b. The applied field is assumed to be uniform with a value of $\boldsymbol{H}_a = -H_o \boldsymbol{e}_x$ and the magnetic permeability of the pipe metal is μ_1, everywhere else the magnetic permeability is assumed to be that of free space $\mu_0 = 4\pi \cdot 10^{-7}$. Laplace's equation can be separated into three ordinary differential equations in (r, ϕ, z) to solve this problem.

3.1 Solution of an infinite hollow cylinder in a uniform field

In cylindrical coordinates if we assume that ψ can be separated into the product of three functions, one for each cylindrical coordinate i.e $\psi = R(r)\,P(\phi)\,Z(z)$, equation (4) will separate into the following ordinary differential equations

$$r^2 \frac{d^2 R}{d r^2} + r \frac{d R}{d r} - \left(k_1^2 + k_2^2\right) R = 0 \tag{13}$$

$$\frac{d^2 P}{d \phi^2} + k_2^2 P = 0 \tag{14}$$

$$\frac{d^2 Z}{d z^2} + k_1^2 Z = 0 \tag{15}$$

where k_1 and k_2 are introduced as separation constants [4, p199]. For the case of an infinite hollow cylinder the potential is independent of z so $k_1 = 0$ and symmetry of the problem dictates that $k_2 = 1$. Solving the above equations for R and P yields a general solution for the scalar magnetic potential given as

$$\psi = \left(c + \frac{d}{r^2}\right)\left(e\sin(\phi) + f\cos(\phi)\right). \tag{16}$$

This equation must be satisfied everywhere, so the values of the c, d, e and f will change depending on the region. Satisfying the requirement that the normal component of $\boldsymbol{B}$ and the tangential component of $\boldsymbol{H}$ must be conserved across both boundaries leads to the solution for the scalar magnetic potential outside the pipeline (subscript 0), inside the metal (subscript 1) and inside the pipeline (subscript 2). We take a closer look at

[1] For an infinite pipe the component of the applied field parallel to the pipeline will induce an anomaly of equal and opposite strength which must be added to the result given here.

the expressions inside and outside the pipe and refer the reader to the appendix for the expressions inside the metal.

3.2 Inside the pipe

Inside the pipe the potential is given as

$$\psi_2 = K\, r\, H_o \cos(\phi)\,, \tag{17}$$

recognizing the applied potential (now in in cylindrical coordinates) as $\psi_a = r\, H_o \cos(\phi)$ we see that the potential inside the pipe is K times the applied potential where K is a geometric constant related to the radii and the permeabilities. It is worth noting that when $\mu_1 \gg \mu_0$, $K \to 0$ and the potential inside is zero, but the anomaly potential defined as $\psi_2^* = \psi_2 - \psi_a \to -\psi_a$. On the other hand when $\mu_1 = \mu_0$, $K \to 1$, $\psi_2 = \psi_a$ and the anomaly $\psi_2^* = 0$.

3.3 Outside the pipe

Outside of the pipeline the potential is

$$\psi_0 = \left(1 + \frac{d_0}{r^2}\right) r\, H_o \cos(\phi)\,. \tag{18}$$

We observe for large r when the effect of the pipe is no longer seen the potential converges to the applied potential. Also, when $\mu_1 \gg \mu_0$, we have $d_0 \to -b^2$ since $\psi_0 = 0$ on the outer radius. (It should be noted the constant d_0 has been scaled from its definition above.)

The anomaly potential outside of the pipeline is

$$\psi_0^* = \frac{d_0}{r^2}\, r\, H_o \cos(\phi) \tag{19}$$

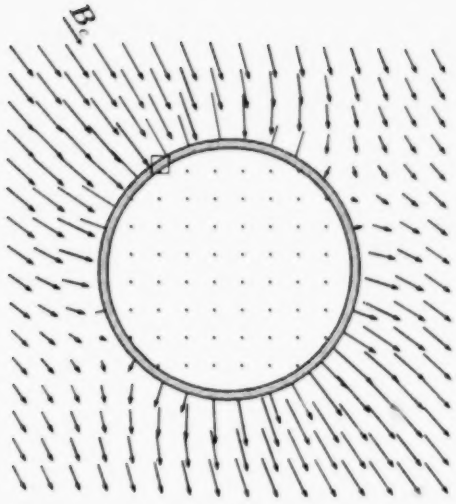
which is of interest for this analysis.

Figure 1 provides a pictorial representation of the magnetic field $\boldsymbol{B}$ and the magnetic anomaly $\boldsymbol{B}^* = \boldsymbol{B} - \boldsymbol{B}_e$ around a pipeline immersed in the earth's magnetic field $\boldsymbol{B}_e$ and the applied field is $\boldsymbol{H}_a = \boldsymbol{B}_e/\mu_0$. We have allowed the applied field to have an inclination from the horizontal typical of the Sable Island gas pipeline region and have used $b = 0.3$m, $a = 0.28$m and a relative permeability of 1000 for the metal.

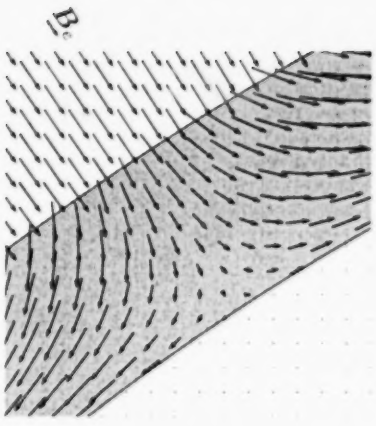
4 Corrosion related electric field

The first line of defence against corrosion is to protect the pipeline with coatings of paint or epoxy, secondary corrosion protection is achieved using either an active systems or passively using sacrificial anodes. When laid in an ocean environment a combination is sometimes used with active systems used near shore and anodes used further out to sea.

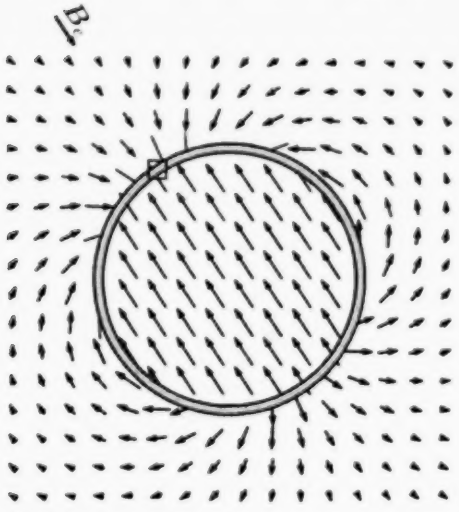
Modelling of an underground pipeline by point current sources was investigated in [6]. For an ocean environment we will represent each anode (active or passive) as a



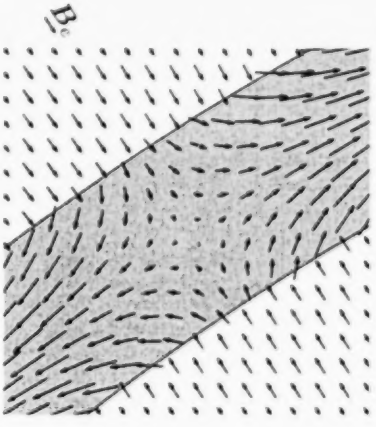
(a) Magnetic field around the pipeline.



(b) Magnetic field near the boundaries.



(c) Magnetic field anomaly around the pipeline.



(d) Magnetic field anomaly near the boundaries.

Figure 1: The induced magnetic field $\boldsymbol{B}$ and the magnetic field anomaly $\boldsymbol{B}^*$ for a pipeline immersed in a uniform field $\boldsymbol{B}_e$ having a magnitude of 50000nT and an inclination from the horizontal of 56.5°. Figures (b) and (d) provide a zoom view of the boxed area in figures (a) and (c), to account for the increased field strength in the metal a different scale is used (note the change in the size of $\boldsymbol{B}_e$)

point current source but will treat the pipeline as a line current source. For this cursory investigation we ignore the boundary effects at the sea surface and sea bottom and consider the pipeline to be in an infinite conducting medium. We will look first at the electric field of two point current sources separated by some distance, the extended electric dipole.

4.1 Extended electric dipole

In a conducting medium point sources of electric current[2] can be used to satisfy Laplace's equation. The electric potential of such a source can be represented as

$$V = \frac{I}{4\pi\sigma R} \tag{20}$$

where R is the distance from the point electric source to the observation point, I is the current flowing from the source and σ is the conductivity of the medium in (Ω^{-1}/m). Such sources are also considered in geophysics exploration using resistivity methods [7]. If we place a positive current source at $z = b$ and a negative current source at $z = a$ (although not required we impose $b > a$), the sum of the two potentials is the potential of an extended electric dipole

$$V = V_b - V_a = \frac{I}{4\pi\sigma}\left(\frac{1}{R_b} - \frac{1}{R_a}\right) \tag{21}$$

where

$$R_a = \sqrt{r^2 + (z-a)^2} \qquad \text{and} \qquad R_b = \sqrt{r^2 + (z-b)^2} \tag{22}$$

see Figure 2. The electric field of the extended dipole is given as $\boldsymbol{E} = -\boldsymbol{\nabla} V$, however using implicit differentiation the required partial derivatives can be calculated in terms of R_a and R_b yielding

$$\boldsymbol{E} = \frac{I}{4\pi\sigma}\left\{\left(\frac{r}{R_b^3} - \frac{r}{R_a^3}\right)\boldsymbol{e}_r + \left(\frac{z_b}{R_b^3} - \frac{z_a}{R_a^3}\right)\boldsymbol{e}_z\right\}, \tag{23}$$

where we further define $z_a = z - a$ and $z_b = z - b$. We have purposely defined the problem so that all the current leaving the positive current source returns to the negative current source.

4.2 Electric field of a line current source

Through the use of the superposition principle differential point current sources can be distributed to form a line current source. If we assume that the distribution is along the z-axis from $z = a$ to $z = b$ and that each current element contributes $di = \rho dz$ of current where ρ is the current line density in (A/m), the voltage for a line current source can be found as

$$V_l = \frac{\rho}{4\pi\sigma}\int_{z'=a}^{b} \frac{dz'}{R_{z'}} \tag{24}$$

[2]This is analogous to a point charge used in electrostatics where $I/\sigma \to q/\epsilon$.

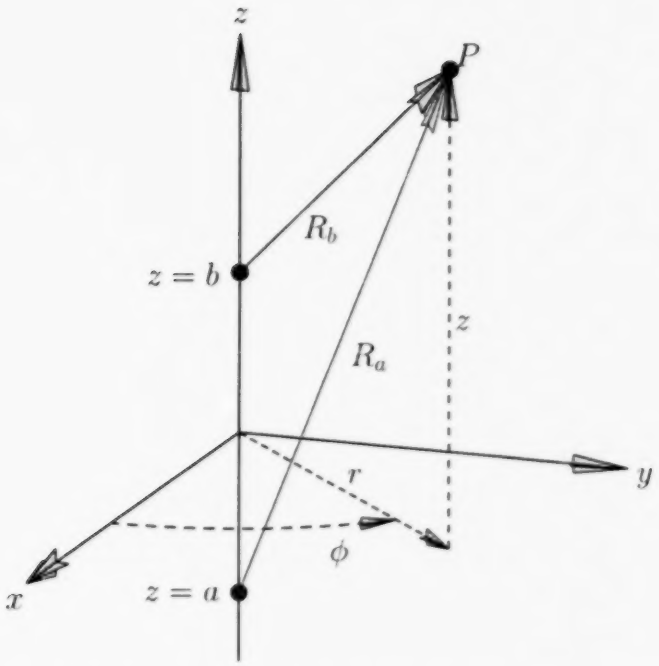


Figure 2: Using the cylindrical coordinates (r, ϕ, z) the distance from the point charge at $z = a$ to the observation point P is $R_a = \sqrt{r^2 + (z-a)^2}$ similarly for the charge at $z = b$, $R_b = \sqrt{r^2 + (z-b)^2}$. The plot is drawn with $b > a$ with $a = -b$.

where z' is the position of the current source element and the distance to the observation point is

$$R_{z'} = \sqrt{r^2 + (z - z')^2}. \tag{25}$$

Evaluation of the integral gives

$$V_l = \frac{\rho}{4\pi\sigma} \left(\ln\left(R_b - z_b\right) - \ln\left(R_a - z_a\right) \right). \tag{26}$$

The electric field for a current line source using implicit differentiation is

$$\boldsymbol{E} = \frac{1}{4\pi\sigma} \left\{ \left(-\frac{\rho r}{R_b\left(R_b - z_b\right)} + \frac{\rho r}{R_a\left(R_a - z_a\right)} \right) \boldsymbol{e}_r + \left(\frac{\rho}{R_b} - \frac{\rho}{R_a} \right) \boldsymbol{e}_z \right\}. \tag{27}$$

which can be assigned either a positive or negative current flow as required.

4.3 Electric field of a line current source and a point current source

Superimposing a line current source and point current source where the line source runs from $z = a$ to $z = b$ and the point source is located at $z = p$ anywhere on the z-axis we

have

$$V = V_p - V_l \tag{28}$$

$$= \frac{1}{4\pi\sigma}\left(\frac{I}{R_p} - \rho \ln\left(R_b - z_b\right) + \rho \ln\left(R_a - z_a\right)\right) \tag{29}$$

and for the electric field

$$\boldsymbol{E} = \frac{1}{4\pi\sigma}\left\{\left(\frac{I\,r}{R_p^3} + \frac{\rho\,r}{R_b\left(R_b - z_b\right)} - \frac{\rho\,r}{R_a\left(R_a - z_a\right)}\right)\boldsymbol{e}_r + \left(\frac{I_n\,z_p}{R_{p_n}^3} - \frac{\rho}{R_b} + \frac{\rho}{R_a}\right)\boldsymbol{e}_z\right\}. \tag{30}$$

To obey conservation of current we require for the total current

$$I = \rho \int_{z'=a}^{b} dz' = \rho\left(b - a\right) = \rho L \tag{31}$$

where $L = b - a$ is the length of the line.

4.4 Electric field of a pipeline

If we assume that there are N anodes attached to a pipeline, a possible static electric model for the pipeline is a line current source with N point current sources. This is a reasonable model if the current demand of the pipeline is uniform (multiple line sources with varying line current densities could also be used) and the field point calculations are re[illegible] for distances considerably greater than the pipe radius. Using the expressions al[illegible] potential for the pipeline with N anodes can be written as

$$V = \left(\sum_{n=1}^{N} V_{p_n}\right) - V_l \tag{32}$$

$$= \frac{1}{4\pi\sigma}\left\{\left(\sum_{n=1}^{N} \frac{I_n}{R_{p_n}}\right) - \rho \ln\left(R_b - z_b\right) + \rho \ln\left(R_a - z_a\right)\right\} \tag{33}$$

where the pipeline runs from $z = a$ to $z = b$, the current of each anode is I_n and the anodes are positioned at $z = p_n$. Conservation of current requires that the total current from the anodes must return to the line, so

$$I = \rho L = \sum_{n=1}^{N} I_n. \tag{34}$$

Note that each anode can have a unique current value if desired. The electric field is the negative gradient of the voltage

$$\boldsymbol{E} = \frac{1}{4\pi\sigma}\left\{\left[\left(\sum_{n=1}^{N} \frac{I_n\,r}{R_{p_n}^3}\right) + \frac{\rho\,r}{R_b\left(R_b - z_b\right)} - \frac{\rho\,r}{R_a\left(R_a - z_a\right)}\right]\boldsymbol{e}_r + \left[\left(\sum_{n=1}^{N} \frac{I_n\,z_p}{R_{p_n}^3}\right) - \frac{\rho}{R_b} + \frac{\rho}{R_a}\right]\boldsymbol{e}_z\right\} \tag{35}$$

5 Corrosion related magnetic field

When electrical currents flow, additional magnetic fields will be produced which must satisfy

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}. \tag{36}$$

To determine $\boldsymbol{H}$ (and $\boldsymbol{B}$) one usually introduces the vector magnetic potential $\boldsymbol{A}$ which relates to the magnetic field as

$$\nabla \times \boldsymbol{A} = \boldsymbol{B}, \tag{37}$$

and is determined using the volume integral

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}}{R^3} dv' \tag{38}$$

where the integration is carried out over the entire volume where $\boldsymbol{J}$ exists. An alternative approach is to use current monopole theory [2] to determine the magnetic field.

5.1 Magnetic field of an extended electric dipole

For the extended electric dipole described above no mention is given as to how the two point current sources have come into existence. The essence of the current monopole theory is that there must be a flow of current that returns the current from one current source to the other current source. The path of this current is arbitrary, so to simplify the problem we choose that it runs in a straight wire filament along the z-axis directly from one source to the other. We can now replace the two current charges with a current filament that runs between the positions of the current sources.

As a result the extended electric dipole is now represented as a current filament that runs from $z = a$ to $z = b$. Since the positive current source is at $z = b$ with $b > a$, the current in the filament will flow along the z-axis in the positive direction, opposite to the direction of $\boldsymbol{J}$ ($\boldsymbol{E}$). The magnetic field can be calculated using the Biot-Savart law

$$\begin{aligned} \boldsymbol{B} &= \frac{\mu I}{4\pi} \int_{l'} \frac{d\boldsymbol{l}' \times \boldsymbol{R}}{R} && (39) \\ &= \frac{\mu I}{4\pi} \int_{z'=a}^{b} \frac{r\, dz'}{R_{z'}} \boldsymbol{e}_\phi && (40) \\ &= -\frac{\mu I}{4\pi} \left(\frac{z_b}{r\, R_b} - \frac{z_a}{r\, R_a} \right) \boldsymbol{e}_\phi. && (41) \end{aligned}$$

5.1.1 Electric field derivation

If the current monopole theory is correct the same electric field must be produced as the original two current source model. We have for $\boldsymbol{E}$ (using $\boldsymbol{B} = \mu \boldsymbol{H}$ and $\boldsymbol{J} = \sigma \boldsymbol{E}$) that

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{\mu\sigma} \nabla \times \boldsymbol{B} && (42) \\ &= \frac{I}{4\pi\sigma} \left\{ \left(\frac{r}{R_b^3} - \frac{r}{R_a^3} \right) \boldsymbol{e}_r + \left(\frac{z_b}{R_b^3} - \frac{z_a}{R_a^3} \right) \boldsymbol{e}_z \right\}, && (43) \end{aligned}$$

which is identical to the expression given in (23) as required.

5.2 Magnetic field of a line current source and a point current source

Since the point current source is located on the axis of the line source each differential element of the line source will form an extended electric dipole with the current point source. The magnetic field of the line source is obtained by integrating the differential magnetic field element of an extended dipole over the length of the line.

$$\boldsymbol{B} = \int d\boldsymbol{B} = -\frac{\mu\rho}{4\pi}\int_{z'=a}^{b}\left(\frac{z_p}{r\,R_p} - \frac{z-z'}{r\,R_{z'}}\right)dz'\boldsymbol{e}_\phi \tag{44}$$

$$= -\frac{\mu}{4\pi}\left(\frac{I\,z_p}{r\,R_p} + \frac{\rho R_b}{r} - \frac{\rho R_a}{r}\right)\boldsymbol{e}_\phi \tag{45}$$

5.2.1 Electric field derivation

Verifying the result using

$$\boldsymbol{E} = \frac{1}{\mu\sigma}\boldsymbol{\nabla}\times\boldsymbol{B} \tag{46}$$

$$= \frac{1}{4\pi\sigma}\left\{\left(\frac{I\,r}{R_p^3} + \frac{\rho\,z_b}{r\,R_b} - \frac{\rho\,z_a}{r\,R_a}\right)\boldsymbol{e}_r + \left(\frac{I_n\,z_p}{R_{p_n}^3} - \frac{\rho}{R_b} + \frac{\rho}{R_a}\right)\boldsymbol{e}_z\right\}, \tag{47}$$

which is equivalent to (30), if it is noted that

$$\frac{z_b}{r\,R_b} - \frac{z_a}{r\,R_a} = \frac{r}{R_b\,(R_b - z_b)} - \frac{r}{R_a\,(R_a - z_a)} \tag{48}$$

which was, at least, numerically proven. Since the electric field determined by using the monopole technique is more efficient we rewrite the equation of the electric field of a pipeline as

$$\boldsymbol{E} = \frac{1}{4\pi\sigma}\left\{\left[\left(\sum_{n=1}^{N}\frac{I_n\,r}{R_{p_n}^3}\right) + \frac{\rho\,z_b}{r\,R_b} - \frac{\rho\,z_a}{r\,R_a}\right]\boldsymbol{e}_r + \left[\left(\sum_{n=1}^{N}\frac{I_n\,z_p}{R_{p_n}^3}\right) - \frac{\rho}{R_b} + \frac{\rho}{R_a}\right]\boldsymbol{e}_z\right\}. \tag{49}$$

5.3 Corrosion related magnetic field of a pipeline

Applying the superposition principle again, N anodes can be placed on the line source and the corrosion related magnetic field of a pipeline becomes

$$\boldsymbol{B} = -\frac{\mu}{4\pi}\left\{\left(\sum_{n=1}^{N}\frac{I_n\,z_{p_n}}{r\,R_{p_n}}\right) + \frac{\rho R_b}{r} - \frac{\rho R_a}{r}\right\}\boldsymbol{e}_\phi. \tag{50}$$

5.4 Field plots of the corrosion related fields

To determine the required spacing and size of the anodes required to protect a pipeline corrosion engineers [5] use the following formula to calculate the current demand of a pipeline

$$I = A f J_d, \tag{51}$$

where A is the pipe surface area, f is the coating breakdown factor and J_d is the current density demand for the bare pipeline. In seawater the current density demand for steel is $60 - 170$ mA/m^2 and coating breakdown factors of 5% are used as a worse case. Noting that the surface area is $A = cL$ where c is the circumference of the pipe and L its length the current line density is given as

$$\rho = \frac{I}{L} = cfJ_d \tag{52}$$

For a visualization we use $c = 2\pi\,(0.3)$, $f = 0.05$ and $J_d = 0.1$, which gives $\rho = 0.01$. We position 11 anodes spaced at 80m and the pipe is 880m long ($b = -a = 440$). Figure (3) provides a view of the corrosion related electric and magnetic fields in an area near the center three anodes. Note the electric field lines flowing outward from the anodes to the pipeline and that the magnetic field has no z-component but its direction alternates along the pipeline consisted with the flow of current. This alternating flow causes a cancellation effect in the farfield so evidently the electric and magnetic field will decay rapidly.

6 Comparison of magnetic field strengths

To compare the relative strengths of the permanent, induced and corrosion related magnetic (CRM) fields and to investigate their fall-off rates we plot the magnitude of these magnetic fields using a logarithmic scale over a long distance as shown in Figure (4). Each magnetic field type is plotted using the minimum and maximum expected values of the parameters to provide a range of field vales for each. We see the permanent field has the greatest effect near the pipeline but that it falls off as $1/r^3$. The induced field although smaller falls off as $1/r^2$ and at about 10m is greater than the permanent field. The upper value of the CRM graph represents a current line density for a bare pipeline and we see that the level in this extreme actually rises above the permanent and induced fields for distances between $10 - 30$m. The CRM initially decays as $1/r$ but the falloff increases to $1/r^4$, the field nulls around 200m due to the cancelling effect of adjacent anodes. Although not shown, the corrosion related electric field falls off similar to the CRM.

7 Conclusions

Analytical expressions were derived for the static electric and magnetic fields due to a pipeline. For the permanent field, observations from the Sable Island gas pipeline suggested the use of randomly oriented dipoles. For the induced magnetic field an analytical solution of a hollow cylinder in a uniform field is used. For the static electric field a line source with point current sources for the anodes was derived. Using the theory of current monopoles an analytical expression for the corrosion related magnetic field of the pipeline was given. A comparison of the field strength was provided and the fall-off rates predicted by the models was shown.

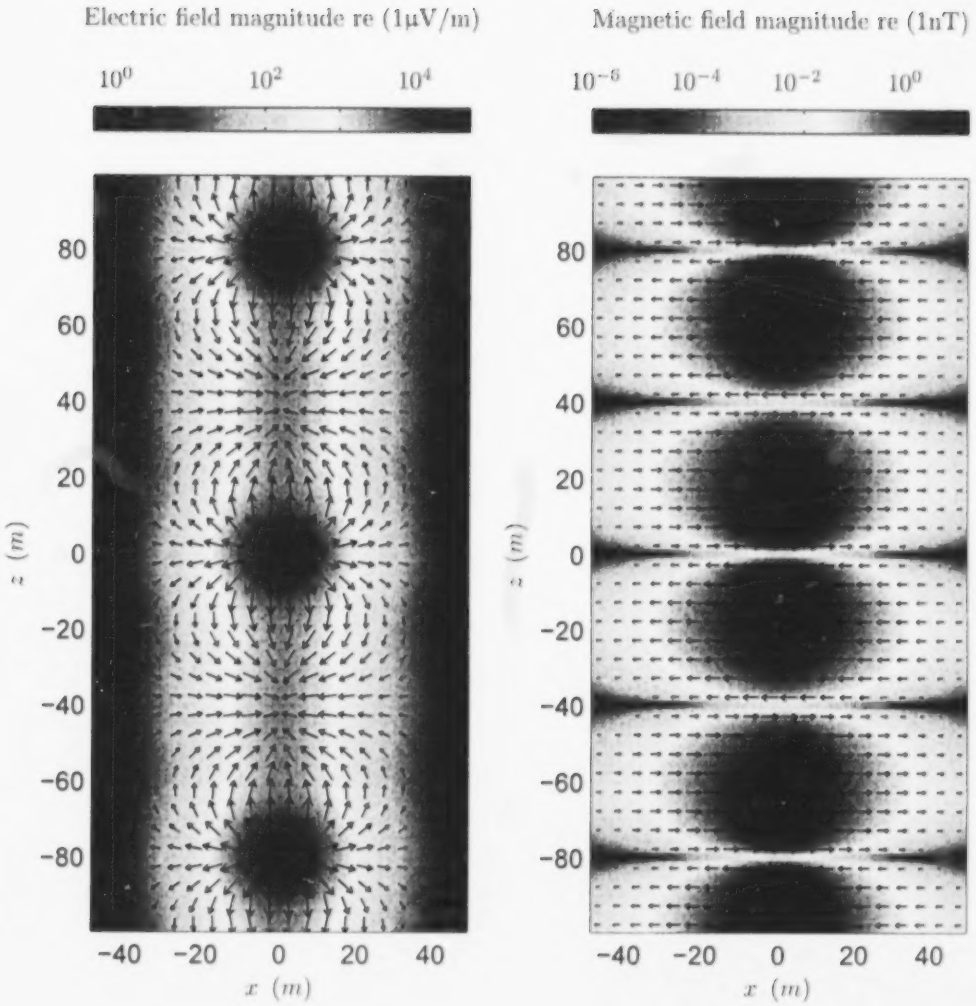


Figure 3: Corrosion related electric and magnetic fields of a pipeline. The current line density is 0.01A/m and the center portion of an 880m pipeline with 11 anodes spaced every 80m is shown. The fields values are calculated on the plane $y = 10$m above the pipeline. The field quivers for x and z properly represent the direction of the field but are logarithmically scaled to enhance the visualization. The colour intensity shows the magnitude of all three components of the fields.

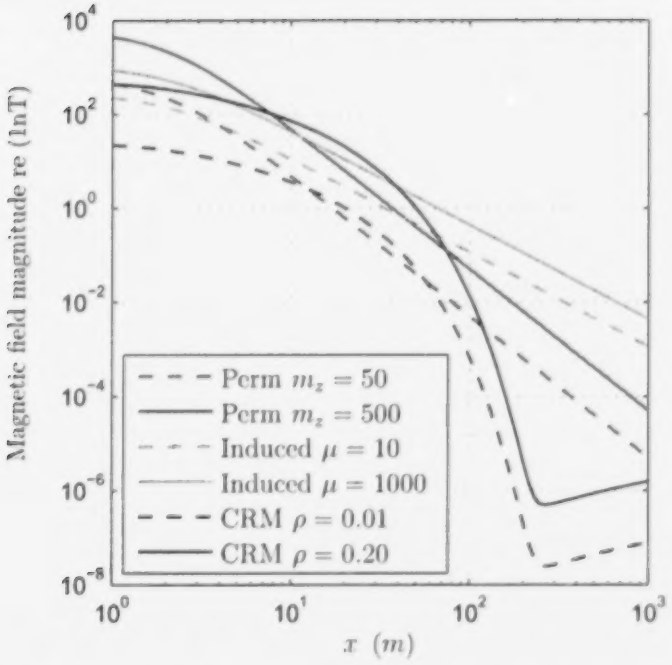


Figure 4: Comparison of the permanent, induced and corrosion related magnetic fields of a pipeline. Each magnetic field type is plotted using minimum and maximum expected values. We see that the permanent field falls-off as $1/r^3$, the induced as $1/r^2$ and the CRM initially as $1/r$ then $1/r^4$.

The development of the corrosion related magnetic model using current monopole theory represents an advancement in the static electromagnetic modelling of pipelines. While the analytically solution derived used point current sources which were constrained to be on the same axis of the line current source this constraint could be lifted to generate a more complex model which could have applications in the corrosion related magnetic modelling of other structures, such as a ship.

Maple's Vector Calculus package was used extensively to derive the expressions and the Maple worksheets are provided as an appendix.

Keywords:
pipeline, static magnetic field, static electric model, induced magnetic field, permanent magnetic field, corrosion related magnetic, current monopoles

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magnetic dipole on z-axis:

in cylindrical coordinates:

```
> restart;
```

```
> with(VectorCalculus):
```

```
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, serie
```

we will define the magnetic moment as:

```
> SetCoordinates(cartesian[x,y,z]);
```

$$cartesian_{x,\,y,\,z}$$

```
> #m := VectorField(<mx,my,mz>);
```

first in cartesian the position vector is:

```
> R := <x,y,z-a>;
```

$$R := x\,e_x + y\,e_y + (z - a)\,e_z$$

the magnitude is:

```
> rr := sqrt(DotProduct(R,R));
```

$$rr := \sqrt{x^2 + y^2 + z^2 - 2\,z\,a + a^2}$$

the potential is:

```
> psi := factor(1/(4*pi)*DotProduct(m,Gradient(1/rr)));
```

$$\psi := \frac{1}{4\,\pi}\left(m \cdot \left(-\frac{x}{(x^2 + y^2 + z^2 - 2\,z\,a + a^2)^{(3/2)}}\,\bar{e}_x - \frac{y}{(x^2 + y^2 + z^2 - 2\,z\,a + a^2)^{(3/2)}}\,\bar{e}_y - \frac{2\,z - 2\,a}{2\,(x^2 + y^2 + z^2 - 2\,z\,a + a^2)^{(3/2)}}\,\bar{e}_z \right) \right)$$

now in cylindrical (m as before):

```
> SetCoordinates(cylindrical[r,phi,z]);
```

$$cylindrical_{r,\,\varphi,\,z}$$

```
> R := <r,phi,z-a>;
```

$$R := r\,e_r + \varphi\,e_\varphi + (z-a)\,e_z$$

```
> _r := simplify(sqrt(DotProduct(R,R)));
```

$$_r := \sqrt{a^2 + z^2 - 2\,z\,a + r^2}$$

```
> psi := factor(1/(4*pi)*DotProduct(m,Gradient(1/_r)));
```

$$\psi := \frac{m \cdot \left(-\frac{r}{(a^2 + z^2 - 2\,z\,a + r^2)^{(3/2)}}\,\bar{e}_r - \frac{2z - 2a}{2\,(a^2 + z^2 - 2\,z\,a + r^2)^{(3/2)}}\,\bar{e}_z \right)}{4\,\pi}$$

```
>
```

Induced Magnetic Field of a Pipeline

```
> restart:
```

```
> with(VectorCalculus):
```

```
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, series
```

Solve Laplace's equation in cylindrical coordinates

```
> SetCoordinates( 'cylindrical'[r,phi,z] );
```

$$cylindrical_{r,\varphi,z}$$

```
> pde1 := Laplacian(phi(r,theta,z)) = 0;
```

$$pde1 := \frac{\left(\frac{\partial}{\partial r}\varphi(r,\theta,z)\right) + r\left(\frac{\partial^2}{\partial r^2}\varphi(r,\theta,z)\right) + r\left(\frac{\partial^2}{\partial z^2}\varphi(r,\theta,z)\right)}{r} = 0$$

```
> alias(R=R(r),P=P(phi),Z=Z(z));
```

$$R, P, Z$$

```
> pde1 := expand(Laplacian(R*P*Z)=0);
```

$$pde1 := \frac{\left(\frac{\partial}{\partial r}R\right) P Z}{r} + \left(\frac{\partial^2}{\partial r^2}R\right) P Z + \frac{R\left(\frac{\partial^2}{\partial \varphi^2}P\right) Z}{r^2} + R P\left(\frac{\partial^2}{\partial z^2}Z\right) = 0$$

```
> pde1 := collect(pde1/(R*P*Z),[R,P,Z],simplify);
```

$$pde1 := \frac{\frac{\partial^2}{\partial z^2}Z}{Z} + \frac{\frac{\partial^2}{\partial \varphi^2}P}{r^2 P} + \frac{\left(\frac{\partial}{\partial r}R\right) + \left(\frac{\partial^2}{\partial r^2}R\right) r}{r R} = 0$$

```
> ode1 := diff(Z,z,z)/Z=-k1^2;
```

$$ode1 := \frac{\frac{\partial^2}{\partial z^2} Z}{Z} = -k1^2$$

```
> pde2 := pde1 - ode1;
```

$$pde2 := \frac{\frac{\partial^2}{\partial \varphi^2} P}{r^2 P} + \frac{\left(\frac{\partial}{\partial r} R\right) + \left(\frac{\partial^2}{\partial r^2} R\right) r}{r R} = k1^2$$

```
> pde2 := expand(simplify(r^2*pde2));
```

$$pde2 := \frac{\frac{\partial^2}{\partial \varphi^2} P}{P} + \frac{r\left(\frac{\partial}{\partial r} R\right)}{R} + \frac{r^2\left(\frac{\partial^2}{\partial r^2} R\right)}{R} = r^2 k1^2$$

```
> ode2 := diff(P,phi,phi)/P=-k2^2;
```

$$ode2 := \frac{\frac{\partial^2}{\partial \varphi^2} P}{P} = -k2^2$$

```
> ode3 := pde2 - ode2;
```

$$ode3 := \frac{r\left(\frac{\partial}{\partial r} R\right)}{R} + \frac{r^2\left(\frac{\partial^2}{\partial r^2} R\right)}{R} = r^2 k1^2 + k2^2$$

```
> ode1 := Z*ode1: ode1 := lhs(ode1)-rhs(ode1) = 0;
```

$$ode1 := \left(\frac{\partial^2}{\partial z^2} Z\right) + Z k1^2 = 0$$

```
> ode2 := P*ode2: ode2 := lhs(ode2)-rhs(ode2) = 0;
```

$$ode2 := \left(\frac{\partial^2}{\partial \varphi^2} P\right) + P k2^2 = 0$$

```
> ode3 := R*ode3: ode3 := simplify(lhs(ode3)-rhs(ode3)) = 0;
```

$$ode3 := r\left(\frac{\partial}{\partial r} R\right) + r^2\left(\frac{\partial^2}{\partial r^2} R\right) - R\,r^2\,k1^2 - R\,k2^2 = 0$$

```
> soln1 := dsolve(ode1);
```

$$soln1 := Z = _C1 \sin(k1\ z) + _C2 \cos(k1\ z)$$

```
> soln2 := dsolve(ode2);
```

$$soln2 := P = _C1 \sin(k2\ \varphi) + _C2 \cos(k2\ \varphi)$$

```
> soln3 := simplify(dsolve(ode3));
```

$$soln3 := R = _C1\ \mathrm{BesselI}(k2, k1\ r) + _C2\ \mathrm{BesselK}(k2, k1\ r)$$

for the case of the infinite pipe psi is independent of z and k1=0:

```
> soln1 := simplify(subs(k1=0,soln1));
```

$$soln1 := Z = _C2$$

```
> soln2;
```

$$P = _C1 \sin(k2\ \varphi) + _C2 \cos(k2\ \varphi)$$

```
> soln3 := dsolve(subs(k1=0,ode3));
```

$$soln3 := R = _C1\ r^{(-k2)} + _C2\ r^{k2}$$

recombining the constants, the general solution is:

```
> soln :=
  subs(subs(_C1=e,_C2=f,soln2),subs(_C1=c,_C2=d,soln3),psi=R*P
  ;
```

$$soln := \psi = (c\ r^{(-k2)} + d\ r^{k2})\ (e \sin(k2\ \varphi) + f \cos(k2\ \varphi))$$

```
> assign(soln);
> 'psi' = psi;
```

$$\psi = (c\ r^{(-k2)} + d\ r^{k2})\ (e \sin(k2\ \varphi) + f \cos(k2\ \varphi))$$

sometimes the constants are reversed so make sure psi in the following form:

```
> psi := (c*r^k2+d*r^(-k2))*(e*sin(k2*phi)+f*cos(k2*phi));
```

$$\psi := (c\ r^{k2} + d\ r^{(-k2)})\ (e \sin(k2\ \varphi) + f \cos(k2\ \varphi))$$

Solve the problem of a cylinder in a uniform field

assume the applied field Ha has a constant value of -Ho in the x-direction:

```
> Ha := VectorField(<-Ho,0,0>,cartesian[x,y,z]);
```

$$Ha := -Ho\,\bar{e}_x$$

map to cylindrical:

```
> Ha := MapToBasis(Ha,cylindrical[r,phi,z]);
```

$$Ha := -Ho\cos(\varphi)\,\bar{e}_r + Ho\sin(\varphi)\,\bar{e}_\varphi$$

it is clear that only k2=1 is required for the perfectly round pipe:

outside the pipe the potential is:

```
> psi0 := subs(e=e0,f=f0,c=c0,d=d0,k2=1,psi);
```

$$\psi 0 := \left(c0\,r + \frac{d0}{r}\right)(e0\sin(\varphi) + f0\cos(\varphi))$$

inside the metal the potential is:

```
> psi1 := subs(e=e1,f=f1,c=c1,d=d1,k2=1,psi);
```

$$\psi 1 := \left(c1\,r + \frac{d1}{r}\right)(e1\sin(\varphi) + f1\cos(\varphi))$$

and interior to the pipe the potential is:

```
> psi2 := subs(e=e2,f=f2,c=c2,d=d2,k2=1,psi);
```

$$\psi 2 := \left(c2\,r + \frac{d2}{r}\right)(e2\sin(\varphi) + f2\cos(\varphi))$$

from the symmetry of the problem we have:

```
> e0:=0: f0:=1: e1:=0: f1:=1: e2:=0: f2:=1:
```

the potential on the interior must remain finite so:

```
> d2:=0:
```

the field interior to the pipe is H2 = -grad(psi2) = K*Ha (some constant times the applied field) so;

```
> H2 := -(Gradient(psi2));
```

$$H2 := -c2\cos(\varphi)\,\bar{e}_r + c2\sin(\varphi)\,\bar{e}_\varphi$$

```
> c2 := solve(K*Ha[1] = H2[1],c2);
```

$$c2 := K\,Ho$$

```
> 'psi2' = psi2;
```

$$\psi 2 = K\,Ho\,r\cos(\varphi)$$

```
> 'H2' = H2;
```

$$H2 = -K\,Ho\cos(\varphi)\,\bar{e}_r + K\,Ho\sin(\varphi)\,\bar{e}_\varphi$$

at distances far from the pipe the field should converge to the applied field:

```
> H0 := -Gradient(psi0);
```

$$H0 := -\left(c0 - \frac{d0}{r^2}\right)\cos(\varphi)\,\bar{e}_r + \frac{\left(c0\,r + \frac{d0}{r}\right)\sin(\varphi)}{r}\,\bar{e}_\varphi$$

```
> c0 := solve(limit(H0[1],r=infinity)=Ha[1],c0);
```

$$c0 := Ho$$

redefine d0 := d0*Ho:

```
> psi0 := subs(d0=d0*Ho,psi0);
```

$$\psi 0 := \left(Ho\,r + \frac{d0\,Ho}{r}\right)\cos(\varphi)$$

```
> H0 := subs(d0=d0*Ho,H0);
```

$$H0 := -\left(Ho - \frac{d0\,Ho}{r^2}\right)\cos(\varphi)\,\bar{e}_r + \frac{\left(Ho\,r + \frac{d0\,Ho}{r}\right)\sin(\varphi)}{r}\,\bar{e}_\varphi$$

redefine c1 := c1*Ho: and d1 := d1*Ho:

```
> psi1 := subs(c1=c1*Ho,d1=d1*Ho,psi1);
```

$$\psi 1 := \left(c1\,Ho\,r + \frac{d1\,Ho}{r}\right)\cos(\varphi)$$

```
> H1 := -Gradient(psi1);
```

$$H1 := -\left(c1\,Ho - \frac{d1\,Ho}{r^2}\right)\cos(\varphi)\,\bar{e}_r + \frac{\left(c1\,Ho\,r + \frac{d1\,Ho}{r}\right)\sin(\varphi)}{r}\,\bar{e}_\varphi$$

the potential of the applied field is:

```
> psi3 := subs(d0=0,psi0);
```

$$\psi 3 := Ho\,r\cos(\varphi)$$

```
> Ha = -Gradient(psi3);
```

$$-Ho\cos(\varphi)\,\bar{e}_r + Ho\sin(\varphi)\,\bar{e}_\varphi = -Ho\cos(\varphi)\,\bar{e}_r + Ho\sin(\varphi)\,\bar{e}_\varphi$$

```
> simplify(MapToBasis(Ha,cartesian[x,y,z]));
```

$$-Ho\,\bar{e}_x$$

the tangential component of H is conserved at the boundaries r=a and r=b, the outside radius = b

```
> eq1 := subs(r=b,H0[2]) = subs(r=b,H1[2]);
```

$$eq1 := \frac{\left(Ho\,b + \frac{d0\,Ho}{b}\right)\sin(\varphi)}{b} = \frac{\left(c1\,Ho\,b + \frac{d1\,Ho}{b}\right)\sin(\varphi)}{b}$$

```
> eq1 := simplify(eq1*b^2/sin(phi));
```

$$eq1 := Ho\,(b^2 + d0) = Ho\,(c1\,b^2 + d1)$$

```
> eq2 := subs(r=a,H1[2]) = subs(r=a,H2[2]);
```

$$eq2 := \frac{\left(c1\,Ho\,a + \frac{d1\,Ho}{a}\right)\sin(\varphi)}{a} = K\,Ho\sin(\varphi)$$

```
> eq2 := simplify(eq2*a^2/sin(phi));
```

$$eq2 := Ho\,(c1\,a^2 + d1) = a^2\,K\,Ho$$

the normal component of B is conserved at the boundaries r=a and r=b:

```
> eq3 := subs(r=b,mu0*H0[1]) = subs(r=b,mu1*H1[1]);
```

$$eq3 := -\mu0\left(Ho - \frac{d0\,Ho}{b^2}\right)\cos(\varphi) = -\mu1\left(c1\,Ho - \frac{d1\,Ho}{b^2}\right)\cos(\varphi)$$

```
> eq3 := simplify(eq3*b^2/cos(phi));
```

$$eq3 := -\mu0\,Ho\,(b^2 - d0) = -\mu1\,Ho\,(c1\,b^2 - d1)$$

```
> eq4 := subs(r=a,mu1*H1[1]) = subs(r=a,mu0*H2[1]);
```

$$eq4 := -\mu1\left(c1\,Ho - \frac{d1\,Ho}{a^2}\right)\cos(\varphi) = -\mu0\,K\,Ho\cos(\varphi)$$

```
> eq4 := simplify(eq4*a^2/cos(phi));
```

$$eq4 := -\mu1\,Ho\,(c1\,a^2 - d1) = -a^2\,\mu0\,K\,Ho$$

```
> soln := solve({eq1,eq2,eq3,eq4},{d0,c1,d1,K}):
```

```
> soln := simplify(subs(subs(mu1=mu*mu0,soln))):
> soln := collect(soln,[a,b],factor);
```

$$soln := \left\{ c1 = -\frac{2(\mu+1)b^2}{(\mu-1)^2 a^2 - (\mu+1)^2 b^2},\ d1 = -\frac{2a^2 b^2 (\mu-1)}{(\mu-1)^2 a^2 - (\mu+1)^2 b^2},\right.$$

$$\left. K = -\frac{4\mu b^2}{(\mu-1)^2 a^2 - (\mu+1)^2 b^2},\ d0 = -\frac{b^2((\mu-1)(\mu+1)a^2 - (\mu-1)(\mu+1)b^2)}{(\mu-1)^2 a^2 - (\mu+1)^2 b^2} \right\}$$

Summary:

```
> vc := Vector(<x,y,z>,'attributes'=['coords'='cartesian'] );
```

$$vc := x\,e_x + y\,e_y + z\,e_z$$

inputs x,y in terms of r and phi:

```
> v := MapToBasis(vc,cylindrical[r,phi,z]);
```

$$v := \sqrt{x^2 + y^2}\ e_r + \arctan(y, x)\, e_\varphi + z\, e_z$$

potentials are:

```
> 'psi0' = psi0; 'psi1' = psi1; 'psi2' = psi2;
```

$$\psi 0 = \left(Ho\, r + \frac{d0\, Ho}{r} \right) \cos(\varphi)$$

$$\psi 1 = \left(c1\, Ho\, r + \frac{d1\, Ho}{r} \right) \cos(\varphi)$$

$$\psi 2 = K\, Ho\, r \cos(\varphi)$$

magnetic intensites are:

```
> 'H0' = H0; 'H1' = H1; 'H2' = H2;
```

$$H0 = -\left(Ho - \frac{d0\, Ho}{r^2} \right) \cos(\varphi)\, \bar{e}_r + \frac{\left(Ho\, r + \frac{d0\, Ho}{r} \right) \sin(\varphi)}{r}\, \bar{e}_\varphi$$

$$H1 = -\left(c1\, Ho - \frac{d1\, Ho}{r^2} \right) \cos(\varphi)\, \bar{e}_r + \frac{\left(c1\, Ho\, r + \frac{d1\, Ho}{r} \right) \sin(\varphi)}{r}\, \bar{e}_\varphi$$

$$H2 = -K\,Ho\cos(\varphi)\,\bar{e}_r + K\,Ho\sin(\varphi)\,\bar{e}_\varphi$$

the anomaly potentials are:

```
> collect(simplify(psi0-psi3),[a,b],factor);
```

$$\frac{Ho\cos(\varphi)\,d0}{r}$$

```
> collect(simplify(psi1-psi3),[a,b],factor);
```

$$\frac{Ho\cos(\varphi)\,(c1\,r^2 + d1 - r^2)}{r}$$

```
> collect(simplify(psi2-psi3),[a,b],factor);
```

$$Ho\,r\cos(\varphi)\,(K - 1)$$

the anomaly fields are:

```
> 'H0a' = simplify(H0-Ha); 'H1a' = simplify(H1-Ha); 'H2a' =
  simplify(H2-Ha);
```

$$H0a = \frac{Ho\cos(\varphi)\,d0}{r^2}\,\bar{e}_r + \frac{Ho\sin(\varphi)\,d0}{r^2}\,\bar{e}_\varphi$$

$$H1a = -\frac{Ho\cos(\varphi)\,(c1\,r^2 - d1 - r^2)}{r^2}\,\bar{e}_r + \frac{Ho\sin(\varphi)\,(c1\,r^2 + d1 - r^2)}{r^2}\,\bar{e}_\varphi$$

$$H2a = -Ho\cos(\varphi)\,(K - 1)\,\bar{e}_r + Ho\sin(\varphi)\,(K - 1)\,\bar{e}_\varphi$$

total magnetic fields are:

```
> 'B0' = mu0*H0; 'B1' = mu1*H1; 'B2' = mu0*H2;
```

$$B0 = -\mu0\left(Ho - \frac{d0\,Ho}{r^2}\right)\cos(\varphi)\,\bar{e}_r + \frac{\mu0\left(Ho\,r + \frac{d0\,Ho}{r}\right)\sin(\varphi)}{r}\,\bar{e}_\varphi$$

$$B1 = -\mu1\left(c1\,Ho - \frac{d1\,Ho}{r^2}\right)\cos(\varphi)\,\bar{e}_r + \frac{\mu1\left(c1\,Ho\,r + \frac{d1\,Ho}{r}\right)\sin(\varphi)}{r}\,\bar{e}_\varphi$$

$$B2 = -\mu0\,K\,Ho\cos(\varphi)\,\bar{e}_r + \mu0\,K\,Ho\sin(\varphi)\,\bar{e}_\varphi$$

```
> assign(soln);
```

```
> limit(d0,mu=infinity);
```

$$-h^2$$

```
> limit(K,mu=infinity);
```

$$0$$

```
>
```

map H from cylindrical to cartesain:

```
> H := VectorField(<Hr(r,phi),Hp(r,phi),0>);
```

$$H := Hr(r, \varphi)\,\bar{e}_r + Hp(r, \varphi)\,\bar{e}_\varphi$$

```
> ax := VectorField(<1,0,0>,cartesian[x,y,z]):
> Hx := DotProduct(ax,H);
```

$$Hx := \cos(\varphi)\,Hr(r, \varphi) - \sin(\varphi)\,Hp(r, \varphi)$$

```
> ay := VectorField(<0,1,0>,cartesian[x,y,z]):
> Hy := DotProduct(ay,H);
```

$$Hy := \sin(\varphi)\,Hr(r, \varphi) + \cos(\varphi)\,Hp(r, \varphi)$$

```
>
```

extended electric dipole in conducting medium.mw

```
> restart;
```

```
> with(plots): with(student):
```

```
Warning, the name changecoords has been redefined
```

```
> with(VectorCalculus):
```

```
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, series
```

```
> interface(imaginaryunit=j):
```

```
> assume(r,real,phi,real,z,real,a,real,b,real,r>0,I,real,mu,real)
```

```
> SetCoordinates( cylindrical[r,phi,z] );
```

$$cylindrical_{r,\varphi,z}$$

define distance functions and perform implict differentiation

```
> alias(Ra=Ra(r,z),Rb=Rb(r,z),za=za(z),zb=zb(z));
```

$$Point, Ra, Rb, za, zb$$

```
> eq_Ra2 := Ra^2 = (r^2+(z-a)^2);
```

$$eq_Ra2 := Ra^2 = r^2 + (z - a)^2$$

```
> eq_Rb2 := Rb^2 = (r^2+(z-b)^2);
```

$$eq_Rb2 := Rb^2 = r^2 + (z - b)^2$$

```
> eq_Ra := Ra = sqrt(rhs(eq_Ra2));
```

$$eq_Ra := Ra = \sqrt{r^2 + z^2 - 2\,z\,a + a^2}$$

```
> eq_Rb := Rb = sqrt(rhs(eq_Rb2));
```

$$eq_Rb := Rb = \sqrt{r^2 + z^2 - 2\,z\,b + b^2}$$

```
> eq_za := za^2 = Ra^2-r^2;
```

$$eq_za := za^2 = Ra^2 - r^2$$

```
> eq_zb := zb^2 = Rb^2-r^2;
```

$$eq_zb := zb^2 = Rb^2 - r^2$$

```
> eq_a := a = z-za;
```

$$eq_a := a = z - za$$

```
> eq_b := b = z-zb;
```

$$eq_b := b = z - zb$$

```
> eq1 := diff(Ra,r) = solve(diff(eq_Ra2,r),diff(Ra,r));
```

$$eq1 := \frac{\partial}{\partial r} Ra = \frac{r}{Ra}$$

```
> eq2 := diff(Ra,z) =
  subs(eq_a,solve(diff(eq_Ra2,z),diff(Ra,z)));
```

$$eq2 := \frac{\partial}{\partial z} Ra = \frac{za}{Ra}$$

```
> eq3 := diff(Rb,r) = solve(diff(eq_Rb2,r),diff(Rb,r));
```

$$eq3 := \frac{\partial}{\partial r} Rb = \frac{r}{Rb}$$

```
> eq4 := diff(Rb,z) =
  subs(eq_b,solve(diff(eq_Rb2,z),diff(Rb,z)));
```

$$eq4 := \frac{\partial}{\partial z} Rb = \frac{zb}{Rb}$$

```
> eq5 := diff(za,z) = solve(diff(eq_a,z),diff(za,z));
```

$$eq5 := \frac{\partial}{\partial z} za = 1$$

```
> eq6 := diff(zb,z) = solve(diff(eq_b,z),diff(zb,z));
```

$$eq6 := \frac{\partial}{\partial z} zb = 1$$

determine the electric field

the potential due to the current source at z=a:

```
> Va := I/(4*Pi*sigma)/Ra;
```

$$Va := \frac{I}{4\,\pi\,\sigma\,Ra}$$

the potential due to the current source at z=b:

```
> Vb := I/(4*Pi*sigma)/Rb;
```

$$Vb := \frac{I}{4\,\pi\,\sigma\,Rb}$$

with the source at z=a negative and the source at z=b positive and the total potential is:

```
> V := Vb - Va;
```

$$V := \frac{I}{4\,\pi\,\sigma\,Rb} - \frac{I}{4\,\pi\,\sigma\,Ra}$$

```
> E := -(Gradient(V));
```

$$E := \left(\frac{I\left(\frac{\partial}{\partial r} Rb\right)}{4\,\pi\,\sigma\,Rb^2} - \frac{I\left(\frac{\partial}{\partial r} Ra\right)}{4\,\pi\,\sigma\,Ra^2} \right) \bar{e}_r + \left(\frac{I\left(\frac{\partial}{\partial z} Rb\right)}{4\,\pi\,\sigma\,Rb^2} - \frac{I\left(\frac{\partial}{\partial z} Ra\right)}{4\,\pi\,\sigma\,Ra^2} \right) \bar{e}_z$$

```
> E := subs(eq1,eq2,eq3,eq4,eq_a,eq_b,E);
```

$$E := \left(\frac{I\,r}{4\,\pi\,\sigma\,Rb^3} - \frac{I\,r}{4\,\pi\,\sigma\,Ra^3} \right) \bar{e}_r + \left(\frac{I\,zb}{4\,\pi\,\sigma\,Rb^3} - \frac{I\,za}{4\,\pi\,\sigma\,Ra^3} \right) \bar{e}_z$$

determine magnetic field using current monopole theory

now using current monopole theory the extended electric dipole
can be represented by a line of current from z=a to z=b using the Biot-Savart Law:

```
> i := VectorField(<0,0,I>);
```

$$i := I\,\bar{e}_z$$

```
> R := VectorField(<r,0,z-_z>);
```

$$R := r\,\bar{e}_r + (z - _z)\,\bar{e}_z$$

```
> dBz :=
  simplify(mu/(4*pi)*CrossProduct(i,R)/DotProduct(R,R)^(3/2));
```

$$dBz := \frac{\mu\,I\,r}{4\,(r^2 + z^2 - 2\,z\,_z + _z^2)^{(3/2)}\,\pi}\,\bar{e}_\varphi$$

```
> B := int(dBz,_z=a..b);
```

$$B := \frac{1}{4\sqrt{r^2+z^2-2zb+b^2}\ r\pi\sqrt{r^2+z^2-2za+a^2}}\left(\mu I\left(\sqrt{r^2+z^2-2zb+b^2}\ z - \sqrt{r^2+z^2-2zb+b^2}\ a - \sqrt{r^2+z^2-2za+a^2}\ z + \sqrt{r^2+z^2-2za+a^2}\ b\right)\right)\bar{e}$$

```
> B := simplify(subs(eq_a,eq_b,B));
```

$$B := -\frac{\mu I\left(-\sqrt{r^2+zb^2}\ za + \sqrt{r^2+za^2}\ zb\right)}{4\sqrt{r^2+za^2}\ r\pi\sqrt{r^2+zb^2}}\bar{e}_\varphi$$

```
> B := simplify(subs(eq_za,eq_zb,B));
```

$$B := \frac{\mu I\,(\mathrm{csgn}(Rb)\,Rb\,za - \mathrm{csgn}(Ra)\,Ra\,zb)\,\mathrm{csgn}(Ra)\,\mathrm{csgn}(Rb)}{4\,Ra\,r\,\pi\,Rb}\bar{e}_\varphi$$

```
> B := subs(csgn(Ra)=1,csgn(Rb)=1,B);
```

$$B := \frac{\mu I\,(Rb\,za - Ra\,zb)}{4\,Ra\,r\,\pi\,Rb}\bar{e}_\varphi$$

```
> B := map(expand,B);
```

$$B := \left(\frac{\mu I\,za}{4\,Ra\,r\,\pi} - \frac{\mu I\,zb}{4\,r\,\pi\,Rb}\right)\bar{e}_\varphi$$

confirm that the same electric field is produced

```
> _E := 1/(sigma*mu)*Curl(B);
```

$$_E := \frac{\dfrac{\mu I\,za\left(\dfrac{\partial}{\partial z}Ra\right)}{4\,Ra^2\,r\,\pi} - \dfrac{\mu I\left(\dfrac{\partial}{\partial z}za\right)}{4\,Ra\,r\,\pi} - \dfrac{\mu I\,zb\left(\dfrac{\partial}{\partial z}Rb\right)}{4\,r\,\pi\,Rb^2} + \dfrac{\mu I\left(\dfrac{\partial}{\partial z}zb\right)}{4\,r\,\pi\,Rb}}{\sigma\,\mu}\bar{e}_r + \frac{1}{\sigma\,\mu}\left(\frac{\mu I\,za}{4\,Ra\,r\,\pi} - \frac{\mu I\,zb}{4\,r\,\pi\,Rb} + r\left(-\frac{\mu I\,za\left(\dfrac{\partial}{\partial r}Ra\right)}{4\,Ra^2\,r\,\pi} - \frac{\mu I\,za}{4\,Ra\,r^2\,\pi} + \frac{\mu I\,zb}{4\,r^2\,\pi\,Rb} + \frac{\mu I\,zb\left(\dfrac{\partial}{\partial r}Rb\right)}{4\,r\,\pi\,Rb^2}\right)\right)\bar{e}_z$$

```
> _E := subs(eq1,eq2,eq3,eq4,eq5,eq6,_E):
> _E := subs(eq_a,eq_b,_E):
> _E := simplify(subs(za^2=Ra^2-r^2,zb^2=Rb^2-r^2,_E)):
> _E := map(expand,(_E));
```

$$_E := \left(-\frac{I\,r}{4\,\sigma\,Ra^3\,\pi} + \frac{I\,r}{4\,\sigma\,\pi\,Rb^3} \right) \bar{e}_r + \left(-\frac{I\,za}{4\,\sigma\,Ra^3\,\pi} + \frac{I\,zb}{4\,\sigma\,\pi\,Rb^3} \right) \bar{e}_z$$

```
>
```

the same as above.

line and point source in conducting medium.mw

```
> restart;
> with(plots): with(student):
Warning, the name changecoords has been redefined
> with(VectorCalculus):
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, series
> interface(imaginaryunit=j):
> assume(r,real,phi,real,z,real,a,real,b,real,r>0,I,real,mu,real)
> SetCoordinates( cylindrical[r,phi,z] );
```

$$cylindrical_{r,\varphi,z}$$

define distance functions and perform implict differentiation

```
> alias(Rp=Rp(r,z),Ra=Ra(r,z),Rb=Rb(r,z),za=za(z),zb=zb(z),zp=
  p(z));
```

$$Point, Rp, Ra, Rb, za, zb, zp$$

```
> eq_Ra2 := Ra^2 = (r^2+(z-a)^2);
```

$$eq_Ra2 := Ra^2 = r^2 + (z-a)^2$$

```
> eq_Rb2 := Rb^2 = (r^2+(z-b)^2);
```

$$eq_Rb2 := Rb^2 = r^2 + (z-b)^2$$

```
> eq_Rp2 := Rp^2 = (r^2+(z-p)^2);
```

$$eq_Rp2 := Rp^2 = r^2 + (z-p)^2$$

```
> eq_Ra := Ra = sqrt(rhs(eq_Ra2));
```

$$eq_Ra := Ra = \sqrt{r^2 + z^2 - 2\,z\,a + a^2}$$

```
> eq_Rb := Rb = sqrt(rhs(eq_Rb2));
```

$$eq_Rb := Rb = \sqrt{r^2 + z^2 - 2\,z\,b + b^2}$$

```
> eq_Rp := Rp = sqrt(rhs(eq_Rp2));
```

$$eq_Rp := Rp = \sqrt{r^2 + z^2 - 2\,z\,p + p^2}$$

```
> eq_za := za = z-a;
```

$$eq_za := za = z - a$$

```
> eq_zb := zb = z-b;
```

$$eq_zb := zb = z - b$$

```
> eq_zp := zp = z-p;
```

$$eq_zp := zp = z - p$$

```
> eq_a := a = z-za;
```

$$eq_a := a = z - za$$

```
> eq_b := b = z-zb;
```

$$eq_b := b = z - zb$$

```
> eq_p := p = z-zp;
```

$$eq_p := p = z - zp$$

```
> eq_za2 := za^2 = Ra^2-r^2;
```

$$eq_za2 := za^2 = Ra^2 - r^2$$

```
> eq_zb2 := zb^2 = Rb^2-r^2;
```

$$eq_zb2 := zb^2 = Rb^2 - r^2$$

```
> eq_zp2 := zp^2 = Rp^2-r^2;
```

$$eq_zp2 := zp^2 = Rp^2 - r^2$$

```
> eq1 := diff(Ra,r) = solve(diff(eq_Ra2,r),diff(Ra,r));
```

$$eq1 := \frac{\partial}{\partial r} Ra = \frac{r}{Ra}$$

```
> eq2 := diff(Ra,z) =
  subs(eq_a,solve(diff(eq_Ra2,z),diff(Ra,z)));
```

$$eq2 := \frac{\partial}{\partial z} Ra = \frac{za}{Ra}$$

```
> eq3 := diff(Rb,r) = solve(diff(eq_Rb2,r),diff(Rb,r));
```

$$eq3 := \frac{\partial}{\partial r} Rb = \frac{r}{Rb}$$

```
> eq4 := diff(Rb,z) =
  subs(eq_b,solve(diff(eq_Rb2,z),diff(Rb,z)));
```

$$eq4 := \frac{\partial}{\partial z} Rb = \frac{zb}{Rb}$$

```
> eq5 := diff(eq_za,z);
```

$$eq5 := \frac{\partial}{\partial z} za = 1$$

```
> eq6 := diff(eq_zb,z);
```

$$eq6 := \frac{\partial}{\partial z} zb = 1$$

```
> eq7 := diff(Rp,r) = solve(diff(eq_Rp2,r),diff(Rp,r));
```

$$eq7 := \frac{\partial}{\partial r} Rp = \frac{r}{Rp}$$

```
> eq8 := diff(Rp,z) =
  subs(eq_p,solve(diff(eq_Rp2,z),diff(Rp,z)));
```

$$eq8 := \frac{\partial}{\partial z} Rp = \frac{zp}{Rp}$$

```
> eq9 := diff(eq_zp,z);
```

$$eq9 := \frac{\partial}{\partial z} zp = 1$$

determine the electric field

the potential due to point current source at at z=p:

```
> Vp := I/(4*Pi*sigma)/Rp;
```

$$Vp := \frac{I}{4 \pi \sigma Rp}$$

for a line source from z=a to b with a uniform current distribution of rho the voltage potential is:

```
> Vl := rho/(4*Pi*sigma)*Int(1/sqrt(r^2+(z-_z)^2),_z=a..b);
```

$$Vl := \frac{\rho \int_a^b \frac{1}{\sqrt{r^2 + z^2 - 2\,z\,_z + _z^2}}\, d_z}{4\,\pi\,\sigma}$$

```
> Vl := simplify(convert(value(Vl),ln)); # this sometimes
  yields incorrect result
```

$$Vl := -\frac{\rho\,(\ln(-z + a + \sqrt{r^2 + z^2 - 2\,z\,a + a^2}) - \ln(-z + b + \sqrt{r^2 + z^2 - 2\,z\,b + b^2}))}{4\,\pi\,\sigma}$$

```
> vv := simplify(convert(int(1/sqrt(r^2+(z-_z)^2),_z),ln));
```

$$vv := -\ln(r) + \ln(-z + _z + \sqrt{r^2 + z^2 - 2\,z\,_z + _z^2})$$

```
> Vl := rho/(4*Pi*sigma)*(subs(_z=b,vv)-subs(_z=a,vv));
```

$$Vl := \frac{\rho\,(\ln(-z + b + \sqrt{r^2 + z^2 - 2\,z\,b + b^2}) - \ln(-z + a + \sqrt{r^2 + z^2 - 2\,z\,a + a^2}))}{4\,\pi\,\sigma}$$

```
> Vl := simplify(subs(eq_a,eq_b,Vl));
```

$$Vl := \frac{\rho\,(\ln(-zb + \sqrt{r^2 + zb^2}) - \ln(-za + \sqrt{r^2 + za^2}))}{4\,\pi\,\sigma}$$

```
> Vl := simplify(subs(eq_za2,eq_zb2,Vl));
```

$$Vl := \frac{\rho\,(\ln(-zb + \mathrm{csgn}(Rb)\,Rb) - \ln(-za + \mathrm{csgn}(Ra)\,Ra))}{4\,\pi\,\sigma}$$

```
> Vl := subs(csgn(Ra)=1,csgn(Rb)=1,Vl);
```

$$Vl := \frac{\rho\,(\ln(-zb + Rb) - \ln(-za + Ra))}{4\,\pi\,\sigma}$$

the total potential is:

```
> V := Vp - Vl;
```

$$V := \frac{I}{4\,\pi\,\sigma\,Rp} - \frac{\rho\,(\ln(-zb + Rb) - \ln(-za + Ra))}{4\,\pi\,\sigma}$$

```
> Ep := -(Gradient(Vp));
```

$$Ep := \frac{I\left(\frac{\partial}{\partial r} Rp\right)}{4\,\pi\,\sigma\,Rp^2}\,\bar{e}_r + \frac{I\left(\frac{\partial}{\partial z} Rp\right)}{4\,\pi\,\sigma\,Rp^2}\,\bar{e}_z$$

```
> Ep := subs(eq7,eq8,Ep);
```

$$Ep := \frac{I\,r}{4\,\pi\,\sigma\,Rp^3}\,\bar{e}_r + \frac{I\,zp}{4\,\pi\,\sigma\,Rp^3}\,\bar{e}_z$$

```
> El := -(Gradient(Vl));
```

$$El := -\frac{\rho\left(\frac{\frac{\partial}{\partial r} Rb}{-zb + Rb} - \frac{\frac{\partial}{\partial r} Ra}{-za + Ra}\right)}{4\,\pi\,\sigma}\,\bar{e}_r - \frac{\rho\left(\frac{-\left(\frac{\partial}{\partial z} zb\right) + \left(\frac{\partial}{\partial z} Rb\right)}{-zb + Rb} - \frac{-\left(\frac{\partial}{\partial z} za\right) + \left(\frac{\partial}{\partial z} Ra\right)}{-za + Ra}\right)}{4\,\pi\,\sigma}\,\bar{e}_z$$

```
> El := subs(eq1,eq2,eq3,eq4,eq5,eq6,El);
```

$$El := -\frac{\rho\left(\frac{r}{Rb\,(-zb + Rb)} - \frac{r}{Ra\,(-za + Ra)}\right)}{4\,\pi\,\sigma}\,\bar{e}_r - \frac{\rho\left(\frac{-1 + \frac{zb}{Rb}}{-zb + Rb} - \frac{-1 + \frac{za}{Ra}}{-za + Ra}\right)}{4\,\pi\,\sigma}\,\bar{e}_z$$

```
> El[3] := simplify(El[3]):
> El := map(expand,El);
```

$$El := \left(-\frac{\rho\,r}{4\,\pi\,\sigma\,Rb\,(-zb + Rb)} + \frac{\rho\,r}{4\,\pi\,\sigma\,Ra\,(-za + Ra)}\right)\bar{e}_r + \left(\frac{\rho}{4\,Rb\,\pi\,\sigma} - \frac{\rho}{4\,Ra\,\pi\,\sigma}\right)\bar{e}_z$$

```
> E := Ep-El;
```

$$E := \left(\frac{I\,r}{4\,\pi\,\sigma\,Rp^3} + \frac{\rho\,r}{4\,\pi\,\sigma\,Rb\,(-zb + Rb)} - \frac{\rho\,r}{4\,\pi\,\sigma\,Ra\,(-za + Ra)}\right)\bar{e}_r + \left(\frac{I\,zp}{4\,\pi\,\sigma\,Rp^3} - \frac{\rho}{4\,Rb\,\pi\,\sigma} + \frac{\rho}{4\,Ra\,\pi\,\sigma}\right)\bar{e}_z$$

determine magnetic field using current monopole theory

now using current monopole theory each current element di will contribute a magnetic field. The amount of field dB is due to an extended electric dipole with current di at positions a=_z and b=p:

recall the result for B for an extended dipole i.e.:

```
> Be := mu*I/(4*Pi)*(-zb/(r*Rb)+za/(r*Ra));
```

$$Be := \frac{\mu I \left(-\frac{zb}{r\,Rb} + \frac{za}{r\,Ra} \right)}{4\,\pi}$$

we place the point source at z=p and the integration value is _z=a to b:

each element has a current of rho*d_z:

```
> B :=
  Int(subs(zb=zp,Rb=Rp,za=z-_z,Ra=sqrt(r^2+(z-_z)^2),I=rho,Be)
  _z=a..b);
```

$$B := \int_a^b \frac{\mu \rho \left(-\frac{zp}{r\,Rp} + \frac{z - _z}{r\sqrt{r^2 + z^2 - 2\,z\,_z + _z^2}} \right)}{4\,\pi} d_z$$

```
> B := value(B):
> B := simplify(subs(eq_a,eq_b,B)):
> B := simplify(subs(eq_za2,eq_zb2,B)):
> B := subs(csgn(Ra)=1,csgn(Rb)=1,eq_za,eq_zb,B):
> B := subs(eq_za,eq_zb,B):
> B := subs(b=I/rho+a,B):
> B := collect(B,{Ra,Rb,Rp},simplify);
```

$$B := \frac{\mu \rho\, Ra}{4\,r\,\pi} - \frac{\mu \rho\, Rb}{4\,r\,\pi} - \frac{\mu\, zp\, I}{4\,r\,\pi\, Rp}$$

```
> B := VectorField(<0,B,0>);
```

$$B := \left(\frac{\mu \rho \, Ra}{4 r \pi} - \frac{\mu \rho \, Rb}{4 r \pi} - \frac{\mu \, zp \, I}{4 r \pi \, Rp} \right) \bar{e}_{\varphi}$$

confirm that the same electric field is produced:

```
> _E := 1/(sigma*mu)*Curl(B);
```

$$_E := \frac{- \frac{\mu \rho \left(\frac{\partial}{\partial z} Ra \right)}{4 r \pi} + \frac{\mu \rho \left(\frac{\partial}{\partial z} Rb \right)}{4 r \pi} + \frac{\mu \left(\frac{\partial}{\partial z} zp \right) I}{4 r \pi \, Rp} - \frac{\mu \, zp \, I \left(\frac{\partial}{\partial z} Rp \right)}{4 r \pi \, Rp^2}}{\sigma \mu} \bar{e}_r + \frac{1}{\sigma \mu} \left(\frac{\mu \rho \, Ra}{4 r \pi} - \frac{\mu \rho \, Rb}{4 r \pi} - \frac{\mu \, zp \, I}{4 r \pi \, Rp} + r \left(- \frac{\mu \rho \, Ra}{4 r^2 \pi} + \frac{\mu \rho \left(\frac{\partial}{\partial r} Ra \right)}{4 r \pi} - \frac{\mu \rho \left(\frac{\partial}{\partial r} Rb \right)}{4 r \pi} + \frac{\mu \rho \, Rb}{4 r^2 \pi} + \frac{\mu \, zp \, I}{4 r^2 \pi \, Rp} + \frac{\mu \, zp \, I \left(\frac{\partial}{\partial r} Rp \right)}{4 r \pi \, Rp^2} \right) \right) \bar{e}_z$$

```
> _E := simplify(subs(eq1,eq2,eq3,eq4,eq7,eq8,eq9,eq_zp2,_E)):
> _E := map(expand,_E);
```

$$_E := \left(- \frac{\rho \, za}{4 \sigma r \pi \, Ra} + \frac{\rho \, zb}{4 \sigma r \pi \, Rb} + \frac{I r}{4 \pi \sigma \, Rp^3} \right) \bar{e}_r + \left(\frac{I \, zp}{4 \pi \sigma \, Rp^3} - \frac{\rho}{4 \, Rb \, \pi \sigma} + \frac{\rho}{4 \, Ra \, \pi \sigma} \right) \bar{e}_z$$

```
> eqr := E[1]=_E[1];
```

$$eqr := \frac{I r}{4 \pi \sigma \, Rp^3} + \frac{\rho \, r}{4 \pi \sigma \, Rb \, (-zb + Rb)} - \frac{\rho \, r}{4 \pi \sigma \, Ra \, (-za + Ra)} = - \frac{\rho \, za}{4 \sigma r \pi \, Ra} + \frac{\rho \, zb}{4 \sigma r \pi \, Rb} + \frac{I r}{4 \pi \sigma \, Rp^3}$$

```
> eqz := E[3]=_E[3];
```

$$eqz := \frac{I \, zp}{4 \pi \sigma \, Rp^3} - \frac{\rho}{4 \, Rb \, \pi \sigma} + \frac{\rho}{4 \, Ra \, \pi \sigma} = \frac{I \, zp}{4 \pi \sigma \, Rp^3} - \frac{\rho}{4 \, Rb \, \pi \sigma} + \frac{\rho}{4 \, Ra \, \pi \sigma}$$

```
> eqr := subs(I=0,eqr);
```

$$eqr := \frac{\rho\, r}{4\,\pi\,\sigma\, Rb\,(-zb + Rb)} - \frac{\rho\, r}{4\,\pi\,\sigma\, Ra\,(-za + Ra)} = -\frac{\rho\, za}{4\,\sigma\, r\,\pi\, Ra} + \frac{\rho\, zb}{4\,\sigma\, r\,\pi\, Rb}$$

```
> eqr := subs(eq_Ra,eq_Rb,eq_za,eq_zb,eqr):
```

```
> evalf(subs(r=10,z=-2,a=5,b=-10,eqr));
```

$$\frac{0.009534633585\,\rho}{\sigma} = \frac{0.009534633572\,\rho}{\sigma}$$

Mapping to cartesian coorinates

map Er and Ez to cartesian:

```
> E1 := VectorField(<Er(r,phi,z),0,Ez(r,phi,z)>);
```

$$E1 := Er(r, \varphi, z)\,\underline{e}_r + Ez(r, \varphi, z)\,\underline{e}_z$$

```
> ax := VectorField(<1,0,0>,cartesian[x,y,z]):
```

```
> Ex := DotProduct(ax,E1);
```

$$Ex := \cos(\varphi)\, Er(r, \varphi, z)$$

```
> ay := VectorField(<0,1,0>,cartesian[x,y,z]):
```

```
> Ey := DotProduct(ay,E1);
```

$$Ey := \sin(\varphi)\, Er(r, \varphi, z)$$

```
> az := VectorField(<0,0,1>,cartesian[x,y,z]):
```

```
> Ezz := DotProduct(az,E1);
```

$$Ezz := Ez(r, \varphi, z)$$

map Bphi to cartesian:

```
> Bp := VectorField(<0,Bphi(r,phi,z),0>);
```

$$Bp := Bphi(r, \varphi, z)\,\underline{e}_\varphi$$

```
> ax := VectorField(<1,0,0>,cartesian[x,y,z]):
```

```
> Bx := DotProduct(ax,Bp);
```

$$Bx := -\sin(\varphi)\, Bphi(r, \varphi, z)$$

```
> ay := VectorField(<0,1,0>,cartesian[x,y,z]):
```

```
> By := DotProduct(ay,Bp);
```

$$By := \cos(\varphi)\, Bphi(r, \varphi, z)$$

```
> az := VectorField(<0,0,1>,cartesian[x,y,z]):
```

```
> Bz := DotProduct(az,Bp);
```

$$Bz := 0$$

```
>
```

TITLE: Static Electromagnetic Fields of an **Underwater Pipeline**

PERSONAL AUTHOR: Richards, T.C.
CORPORATE AUTHOR: Defence R&D Canada - Atlantic, Dartmouth NS (CAN)

ABSTRACT: Analytical expressions for the static electromagnetic fields due to a pipeline are given for the permanent and induced magnetic fields as well as the corrosion related electric and magnetic fields. Randomly oriented dipoles are used for the permanent magnetic field while the induced field is calculated using the solution of a hollow cylinder in a uniform field. For the corrosion related electric field, expressions for a line source with multiple point current sources representing sacrificial anodes is derived. Using current monopole theory the electric model is equivalently represented with current filaments to provide expressions for the corrosion related magnetic field. A comparison of the magnetic field strengths is provided and the fall-off rates predicted by the models is shown. Maple's Vector Calculus library is used extensively in the derivations and Maple worksheets with the full solution details are provided in the appendix.
Il y a un résumé français ici.

IDENTIFIERS: pipeline, static magnetic field, static electric model, induced magnetic field, ;magnetic field, corrosion related magnetic, current monopoles

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Important Notices